J. Appl. Maths Mechs, Vol. 55, No. 4, pp. 469-474, 1991
Printed in Great Britain

0021-8928/91 \$15.00+0.00

ASYMPTOTIC SOLUTIONS OF HAMILTONIAN SYSTEMS WITH FIRST-ORDER RESONANCE*

B.S. BARDIN

Moscow

(Received 25 October 1990)

Solutions that are asymptotic with respect to the equilibrium for a Hamiltonian system with a first-order resonance are considered. The Hamiltonian function is assumed to be analytic in the neighbourhood of the equilibrium and 2π-periodic in time or time-independent. The analysis is carried out by the method described in /1-3/. The case of a Hamiltonian system with one degree of freedom is investigated in detail. For multidimensional Hamiltonian systems, sufficient conditions for the existence of asymptotic solutions and their approximate analytical representation are derived.

1. Consider a Hamiltonian system of ordinary differential equations

$$\frac{dq}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial q} \tag{1.1}$$

Assume that the origin $q=p=0$ is an equilibrium of this system and that the Hamiltonian function H can be represented in a sufficiently small neighbourhood of the origin by a convergent series

$$H=\sum_{k=2}^{\infty} H_k(q,p,t) \tag{1.2}$$

where H_k is a homogeneous form of degree k in q, p with coefficients that are continuous 2π-periodic in t.

We assume that both roots of the characteristic equation of the linearized system (1.1) equal unity, i.e., the case of first-order resonance /4/.

We consider the existence and analytical construction of the solutions of system (1.1) that asymptotically converge to the origin as $t\to\pm\infty$. In Sect.4, the results obtained for system (1.1) are partially extended to multidimensional Hamiltonian systems with a simple first-order resonance.

The classical theory of asymptotic solutions /5, 6/ is applicable only when the linearized system has at least one non-zero characteristic value. The problem of solutions that are asymptotic with respect to the equilibrium of the Hamiltonian system with all zero characteristic values was considered in /1-3/. The corresponding methods are used below.

2. The problem of asymptotic solutions of system (1.1) is investigated for two separate cases. First let us consider the case of simple elementary divisors of the characteristic matrix of the linearized system (1.1). In this case, the Hamiltonian function (1.2) can be reduced by an appropriate change of variables (e.g.,by Birkhoff transformation /7/) to the normal form /4/

$$H=\Phi(\varphi)\,r^{M/2}+H^{(M+1)}(r,\ \varphi,\ t)\ \ (M\geqslant 3) \tag{2.1}$$

$$q=\sqrt{2r}\sin\varphi,\quad p=\sqrt{2r}\cos\varphi,\qquad \Phi(\varphi)=\sum_{j=0}^{M} a_j\sin^j\varphi\cos^{M-j}\varphi \tag{2.2}$$

Here a_j are constant coefficients and $H^{(M+1)}$ are terms of order higher than M in $\sqrt{r}$ and 2π-periodic in t and φ.

Making a non-canonical change of variables $r=\rho^2$, $\varphi=0$, we obtain the equations of motion in the form

*Prikl.Matem.Mekhan.,55,4,587-593,1991

$$\frac{d\theta}{dt} = \frac{M}{2}\rho^{M-2}\Phi(\theta) + P(\rho, \theta, t), \qquad \frac{d\rho}{dt} = \frac{1}{2}\rho^{M-1}F(\theta) + R(\rho, \theta, t) \tag{2.3}$$

where $F(\theta) = -d\Phi(\theta)/\partial\theta$. The functions P and R are analytic in the neighbourhood of the origin and their series expansions in powers of ρ start with terms of degree not lower than $(M-1)$ and M, respectively; P and R are periodic in θ and t and tend to zero uniformly in these variables as $\rho \to 0$.

The theory of analysis of the neighbourhood of a singular point developed in /8/ is applicable to system (2.3). On the basis of this theory, we can conclude, as in /1, 2/, that the trajectories of the system in the plane $x = \rho\cos\theta$, $y = \rho\sin\theta$ may enter the origin only in directions defined by the angles θ_*, where θ_* is the real root of the equation $\Phi(\theta) = 0$. To each simple root θ_* there corresponds a unique integral curve that enters the origin as $t \to +\infty$ (if $F(\theta_*) < 0$) or as $t \to -\infty$ (if $F(\theta_*) > 0$), because in this case we have $F(\theta_*)\,d\Phi(\theta_*)/d\theta < 0$. If θ_* is a multiple root, then the existence of asymptotic trajectories corresponding to this root is decided by terms of order not higher than M in $\sqrt{r}$ in the expansion of the Hamiltonian (2.1).

It can be shown that if θ_* is a real root of the equation $\Phi(\theta) = 0$, then $[\theta_* + \pi]$ (mod 2π) is also a real root of this equation. The number of roots is obviously always even or zero. The maximum number of roots is $2M$.

If the equation $\Phi(\theta) = 0$ has no real roots, then /9/ the point $q = p = 0$ is encircled by invariant curves that pass as close as desired to this point /10, 11/. Therefore, trajectories asymptotic to the origin do not exist, because otherwise the uniqueness of the Cauchy problem would be violated.

We will investigate the analytical structure of the asymptotic solutions using Zubov's results /12/. We will briefly review them here. Consider the system

$$\tau\frac{dy_s}{d\tau} = \sum_{j=1}^{n} a_{sj}(\tau)\, y_j + a_s(\tau)\,\tau + Y_s(\tau, y_1, \ldots, y_n) \tag{2.4}$$

The functions Y_s are expanded in series,

$$Y_s = \sum_{m+m_1+\ldots+m_n \geqslant 2} A_s^{(m, m_1, \ldots, m_n)}(\tau)\,\tau^m y_1^{m_1}\ldots y_n^{m_n}, \quad s = 1, \ldots, n \tag{2.5}$$

that converge for $|\tau| < \tau_1, \tau_1 > 0$ is a constant, $|y_j| < y_0, (j = 1, \ldots, n)$. The functions $a_{sj}(\tau)$, $a_s(\tau)$, $A_s^{(m, m_1, \ldots, m_n)}(\tau)$ are real, continuous, and bounded for $0 < \tau \leqslant 1$.

Denote by $1, \mu_1, \ldots, \mu_n$ the characteristic values of the system

$$\frac{dz}{d\eta} = -z, \qquad \frac{dy_s}{d\eta} = -\sum_{j=1}^{n} a_{sj}(e^{-\eta})\, y_j - a_j(e^{-\eta})\, z \tag{2.6}$$

Theorem /12/. If $\mu_k > 0$ for $k \leqslant l$ and system (2.6) is regular, then the system of Eqs.(2.1) has a family of solutions that depends on l arbitrary constants and can be represented in the form of the series

$$y_s = \sum_{m+m_1+\ldots+m_l \geqslant 1} K^{(m, m_1, \ldots, m_l)}(\tau)\,\tau^{(m+\mu_1 m_1+\ldots+\mu_l m_l)} c_1^{m_1}\ldots c_l^{m_l}, \; s = 1, 2, \ldots, n \tag{2.7}$$

that converge for $|\tau| < \tau_0$, $|c_i| < c_0$ $(i = 1, 2, \ldots, l)$, where τ_0 is a fairly small quantity, $c_0 = \text{const}$, and $K_i^{(m, m_1, \ldots, m_l)}(\tau)\,\tau^\alpha \to 0$ as $\tau \to 0$, where $\alpha > 0$ is a constant.

Consider the question of the analytical construction of asymptotic solutions. Change to Cartesian coordinates by the formulas $x = \rho\cos\theta$, $y = \rho\sin\theta$. The equations of motion in the new variables take the form

$$\frac{dx}{dt} = -\frac{1}{2}\sum_{k=1}^{M} a_k k y^{k-1} x^{M-k} + X(x, y, t) \tag{2.8}$$

$$\frac{dy}{dt} = \frac{1}{2}\sum_{k=0}^{M-1} a_k (M-k)\, y^k x^{M-k-1} + Y(x, y, t)$$

where X and Y are functions that are 2π-periodic in t and analytic in x, y, and their series expansions start with terms of degree not lower than M.

Let us first consider the asymptotic solutions of the truncated system, which is obtained from system (2.8) if the terms $X(x, y, t), Y(x, y, t)$ are omitted on the right-hand sides of its equations. To each simple root of the equation $\Phi(\theta) = 0$ there corresponds a family of asymptotic solutions of the truncated systems

$$x(t) = x(0)\cos\theta_*[\cos^{M-2}\theta_* - {}^1/_2(M-2)F(\theta_*)x^{M-2}(0)t]^{-1/(M-2)}$$
$$y(t) = y(0)\sin\theta_*[\sin^{M-2}\theta_* - {}^1/_2(M-2)F(\theta_*)y^{M-2}(0)t]^{-1/(M-2)} \tag{2.9}$$

We will show that in the full system (2.8) a one-parameter family of asymptotic solutions also corresponds to each simple root θ_*. Given the structure of the solution (2.8) and in order to apply Zubov's theorem /12/ to the problem, we make the change of variables $x, y, t \to y_1, y_2, \tau$ by the formulas

$$x = \tau(y_1 + a),\ y = \tau(y_2 + b),\ \tau = t^{-1/(M-2)}$$
$$a = A\cos\theta_*,\ b = A\sin\theta_*,\ A = [{}^1/_2(M-2)\,|F(\theta_*)|]^{-1/(M-2)} \tag{2.10}$$

In the new variables, system (2.4) takes the form

$$\tau\frac{dy_1}{d\tau} = \Big[\frac{M-2}{2}\sum_{k=1}^{M} a_k k(M-k)a^{M-k-1}b^{k-1} - 1\Big]y_1 +$$
$$\Big[\frac{M-2}{2}\sum_{k=1}^{M} a_k k(k-1)a^{M-k}b^{k-2}\Big]y_2 + \zeta_1 \tag{2.11}$$
$$\tau\frac{dy_2}{d\tau} = \Big[\frac{M-2}{2}\sum_{k=0}^{M-1} a_k(M-k)(M-k-1)b^k a^{M-k-2}\Big]y_1 -$$
$$\Big[\frac{M-2}{2}\sum_{k=0}^{M-1} a_k k(M-k)a^{M-k-1}b^{k-1} + 1\Big]y_2 + \zeta_2$$
$$(\zeta_i = -\tau(M-2)f_i(a, b, \tau^{-1}) + Y_i(y_1, y_2, \tau),\ i = 1, 2$$

where $f_i(x, y, t)$ are terms of degree M in the expansion in powers of x, y of the functions X and Y respectively, and the functions Y_i can be represented by series of the form (2.5).

If Y_1 and Y_2 are omitted on the right-hand sides of system (2.11) and we make the change of independent variable $\eta = -\ln\tau$, then the result is a linear system of the form (2.6). This system is regular and its characteristic values are $1, (M-2), -M$. Thus, by Zubov's theorem we can assert that system (2.11) has a one-parameter family of solutions, which can be represented by series of the form (2.7).

In the variables x, y, t, for sufficiently large $|t|$, we obtain the following representation of asymptotic solutions of system (2.8) that correspond to the root θ_*:

$$x = at^{-1/(M-2)} + \psi(t, c)t^{-2/(M-2)},\quad y = bt^{-1/(M-2)} + \chi(t, c)t^{-2/(M-2)} \tag{2.12}$$

$\psi(t, c)$ and $\chi(t, c)$ are functions of time and the arbitrary constant c and they are uniformly bounded for sufficiently large $|t|$.

If all the roots of the equation $\Phi(\theta) = 0$ are simple, then obviously other asymptotic solutions do not exist.

3. Now let us investigate the problem of asymptotic solutions of system (1.1) for the case of multiple elementary divisors of the characteristic matrix of the linearized system. In this case, the Hamiltonian function normalized to terms of order M in q, p has the form /4, 13/

$$H = H_0 + \sum_{\alpha+\beta>M} h_{\alpha\beta}q^\alpha p^\beta,\quad H_0 = {}^1/_2\delta q^2 + a_{M,0}p^M;\quad \delta = \pm 1 \tag{3.1}$$

where $a_{M,0}$ is a constant real coefficient and $h_{\alpha\beta}(t)$ are continuous 2π-periodic functions of time.

A complete qualitative analysis of the solutions asymptotic to the equilibrium of system (1.1) with the Hamiltonian (3.1) has been carried out in /13/.

The asymptotic solutions of the truncated system with the Hamiltonian H_0 have the form

$$q = d(c \pm t)^{-M/(M-2)},\ p = g(c \pm t)^{-2/(M-2)},\ g = \pm{}^1/_2(M-2)\delta d \tag{3.2}$$

where c is an arbitrary constant and d is the real root of the equation

$$d^{M-2}+(\pm 1)^M 2^{M-1}[(M-2)^M a_{M,0}\delta^{M-1}]^{-1}=0 \tag{3.3}$$

In formulas (3.2), (3.3) the upper sign corresponds to $t\to+\infty$ $(t>-c)$ and the lower sign to $t\to-\infty$ $(t<c)$.

To obtain an analytical representation of the asymptotic solutions of system (1.1), given the structure of the solution (3.2), we make the change of variables

$$q=\tau^M(x+d),\ \ p=\tau^2(y+g),\ \ \tau=t^{-1/(M-2)}$$

In the new variables x, y, τ, the complete system (1.1) with the Hamiltonian (3.1) is

$$\begin{aligned}\tau dx/d\tau &= -Mx-Ma_{M,0}(M-1)(M-2)g^{M-2}y+X_1(x,\ y,\ \tau)\\ \tau dy/d\tau &= -2y+\delta(M-2)x+X_2(x,\ y,\ \tau)\end{aligned} \tag{3.4}$$

where X_i $(i=1,\ 2)$ are functions of type (2.5).

Zubov's theorem can be applied to system (3.4). Indeed, the linear system (2.6) corresponding to (3.4) is regular and its characteristic values are 1, $(M-2)$ and $-2M$. System (3.4) thus has a one-parameter family of asymptotic solutions that can be represented as series of the form (2.7).

In the variables q, p, t, for sufficiently large $|t|$, we obtain the following representation of the asymptotic solutions of (1.1) with the Hamiltonian (3.1), which enter the origin as $t\to+\infty$ as $t\to-\infty$:

$$q=dt^{-M/(M-2)}+\varkappa(t,c)\,t^{-(M+2)/(M-2)},\ \ p=gt^{-2/(M-2)}+\eta(t,c)\,t^{-4/(M-2)} \tag{3.5}$$

where $\varkappa(t,c)$, and $\eta(t,c)$ are functions of time and the arbitrary constant c and they are uniformly bounded for sufficiently large $|t|$.

4. Let us briefly consider the existence and analytical construction of solutions asymptotic to the equilibrium of the Hamiltonian system with n $(n\geqslant 2)$ degrees of freedom and a first-order resonance.

Suppose we are given the Hamiltonian system of differential equations

$$dq_i/dt=\partial H/\partial p_i,\ \ dp_i/dt=-\partial H/\partial q_i\ \ (i=1,\ 2,\ \ldots,\ n) \tag{4.1}$$

and the origin $q_i=p_i=0$ $(i=1,2,\ldots,n)$ is an equilibrium of this system. In the neighbourhood of $q_i=p_i=0$ the Hamiltonian function H is analytic and 2π-periodic in t or independent of t. Consider the problem of the solutions of system (4.1) which are asymptotic to its equilibrium $q_i=p_i=0$. We will assume that all the characteristic exponents $\pm\lambda_k$ of the linearized system have zero real parts. We also assume a first-order resonance in system (4.1), i.e.,

$$\varphi_*,\ r_1=r_1(0)\,\nu^{-2/(} \tag{4.2}$$

where N is an integer (if H is time independent, then $N=0$). Without loss of generality, we may henceforth assume $k=1$.

With an appropriate choice of the variables q_i, p_i (e.g., using the Birkhoff transformation /8/ or the Depri-Khuri method /14/), the Hamiltonian function H may be reduced to the following form /9, 15/:

a) for the case of simple elementary divisors of the characteristic matrix of the linearized system (4.1),

$$H=\sum_{i=2}^{n}\sigma_i r_i+\Phi_{M_1}(\varphi_1)\,r_1^{M_1/2}+\sum_{m=3}^{M_1}\sum_{k=1}^{[m/2]}\sum_{|\alpha|=k}\Phi_{m-2k,\alpha}(\varphi_1)\,r_1^{m/2-k}r_2^{\alpha_2}\ldots r_n^{\alpha_n}+O_{M_1} \tag{4.3}$$

b) for the case of multiple elementary divisors of the characteristic matrix of the linearized system (4.1),

$$H=\frac{\delta}{2}q^2+\sum_{i=2}^{n}\sigma_i r_i+a_{M2}p_1^{M_2}+\sum_{m=3}^{M_2}\sum_{k=1}^{[m/2]}\sum_{|\alpha|=k}a_{m-2k,\alpha}p_1^{m-2k}r_2^{\alpha_2}\ldots r_n^{\alpha_n}+O_{M_2} \tag{4.4}$$

$$q_i=\sqrt{2r_i}\sin\varphi_i,\ \ p_i=\sqrt{2r_i}\cos\varphi_i\ \ (i=1,\ \ldots,\ n),\ \ \delta=\pm 1$$

$$\alpha=(\alpha_2,\ldots,\alpha_n),\quad |\alpha|=\sum_{j=2}^{n}\alpha_j$$

$$\Phi_{m-2k}(\varphi_1)=\sum_{j=1}^{m-2k}b_{j,\alpha}\sin^j\varphi_1\cos^{m-2k-j}\varphi_1$$

$$\Phi_{M_1}(\varphi_1) = \sum_{j=1}^{M_1} b_j \sin^j \varphi_1 \cos^{M_1-j}\varphi_1$$

where a_{M_2}, $a_{m-2k,\alpha}$, b_i and $b_{j,\alpha}$ are real constants; square brackets denote the operation of taking the integer part of the number. We assume that the normalization is carried out for orders M_1 and M_2, such that $\Phi_{M_1}(\varphi_1) \not\equiv 0$ and $a_{M_2} \neq 0$, respectively. We denote by O_{M_1} and O_{M_2} respectively terms of order higher than M_1 and M_2 in q_i, p_i $(i = 1, \ldots, n)$.

We make a canonical change of variables

$$\rho_i = r_i, \ \theta_i = \sigma_i t + \varphi_i \ (i = 2, \ldots, n)$$

In the new canonical variables, the Hamiltonians (4.3) and (4.4) respectively take the form

$$\text{a) } H = \Phi_{M_1}(\varphi_1) r_1^{M_1/2} + \sum_{m=3}^{M_1} \sum_{k=1}^{[m/2]} \sum_{|\alpha|=k} \Phi_{m-2k,\alpha}(\varphi_1) r_1^{m/2-k} \rho_2^{\alpha_2} \cdots \rho_n^{\alpha_n} + O_{M_1} \tag{4.5}$$

$$\text{b) } H = \frac{\delta}{2} q_1^2 + a_{M,0} p_1^{M_2} + \sum_{m=3}^{M_2} \sum_{k=1}^{[m/2]} \sum_{|\alpha|=k} a_{m-2k,\alpha} p^{m-2k} \rho_2^{\alpha_2} \cdots \rho_n^{\alpha_n} + O_{M_2} \tag{4.6}$$

First consider the truncated system formed by omitting in the Hamiltonian functions (4.5) and (4.6) terms of degree higher than M_1 and M_2 in $\sqrt{r_1}, \sqrt{\rho_i}$ and $q_1, p_1, \sqrt{\rho_i}$ $(i = 2, 3, \ldots, n)$, respectively. Direct integration gives the following results.

a) *Simple elementary divisors.* To each simple root φ_* of the equation $\Phi_{M_1}(\varphi_1) = 0$ there corresponds a one-parameter family of solutions asymptotic as $t \to +\infty$ (if $d\Phi_{M_1}(\varphi_*)/d\varphi_1 > 0$) or as $t \to -\infty$ (if $d\Phi_{M_1}(\varphi_*)/d\varphi_1 < 0$) to the origin $q_i = p_i = 0$ $(i = 1, \ldots, n)$:

$$\begin{gathered}\varphi_1 = \varphi_*, \ r_1 = r_1(0)\, \nu^{-2/(M-2)} \\ \nu = 1 + \frac{M_1 - 2}{2} \frac{d\Phi_{M_1}(\varphi_*)}{d\varphi_1} r_1^{(M_1-2)/2}(0)\, t \\ \theta_j = \left[\frac{d\Phi_{M_1}(\varphi_*)}{d\varphi_1}\right]^{-1} \left\{ \sum_{m=3}^{M_1-1} \frac{2}{M_1 - m} r_1^{(m-M_1)/2}(0)\, \Phi_{m-2,\alpha^j}(\varphi_*)\, \nu^{(M_1-m)/(M_1-2)} + \right. \\ \left. \frac{\Phi_{M_1-2,\alpha^j}(\varphi_*) \ln \nu^2}{M_1 - 2} \right\}, \quad \rho_j = 0 \\ (j = 2, \ldots, n)\end{gathered} \tag{4.7}$$

b) *Multiple elementary divisors.* If M_2 is odd, then two one-parameter families of solutions exist (one family as $t \to +\infty$ and one family as $t \to -\infty$) asymptotic to the origin,

$$\begin{gathered}q_1 = d\,(c \pm t)^{-M_2/(M-2)}, \ p_1 = q\,(c \pm t)^{-2/M_2-2)} \\ \theta_j = \sum_{m=3}^{M_2} \frac{M_2 - 2}{M_2 - 2m - 2} a_{m-2,\alpha^j} g^{m-2} l_m(c \pm t) \\ l_m(t) = \begin{cases} t^{(M_2-2m-2)/(M_2-2)}, & M_2 \neq 2m - 2 \\ \ln t, & M_2 = 2m - 2 \end{cases} \\ \rho_j = 0, \ \alpha^j = (0, \ldots, \alpha_j, \ldots, 0), \ \alpha_j = 1 \ (j = 2, \ldots, n)\end{gathered} \tag{4.8}$$

where c is an arbitrary constant and d and g are given by formulas (3.2) and (3.3).

If M_2 is even and $a_{M_2}\delta < 0$, then four one-parameter families of asymptotic solutions exist (two families as $t \to +\infty$ and two families as $t \to -\infty$) of the form (4.8).

For simplicity, let $M_1 = 3, M_2 = 4$. Using the structure of solutions of the truncated system and applying Zubov's theorem, we can prove the existence of asymptotic solutions of the complete system (4.1), which are approximately represented by formulas (4.7) and (4.8). To this end we need to make the change of variables

$$\begin{gathered}\text{a) } \rho_i = \tau^2 y_i, \ \ \theta_i = 2\Phi_{1,\alpha^i}(\varphi_*)\,[(d\Phi_3(\varphi_*))/d\varphi_1]^{-1} \ln \tau^{-1} + x_i \ \ (i = 2, \ldots, n), \\ r_1 = \tau^2 \{4\,[(d\Phi_3(\varphi_*))/d\varphi_1]^{-2} + y_1\} \\ \varphi_1 = \varphi_* + x_1, \ \tau = t^{-1}\end{gathered} \tag{4.9}$$

$$\begin{gathered}\text{b) } \rho_i = \tau^4 y_i, \ \theta_i = g a_{1,\alpha^i} \ln \tau^{-1} + x_i \\ p_1 = \tau^2 (g + y_1), \ q_1 = \tau^4 (d + x_1) \ (i = 2, \ldots, n), \quad \tau = t^{-1/4}\end{gathered} \tag{4.10}$$

where φ_* is the root of the equation $\Phi_3(\varphi_1) = 0$, $\boldsymbol{\alpha}^i = (0, \ldots, \alpha_i, \ldots, 0)$, $\alpha_i = 1$.

Zubov's theorem can be applied to the system obtained by this change of variables, as in Sects.2 and 3.

For $M_1 > 3$ and $M_2 > 4$, the proof of the existence of asymptotic solutions for the complete system (4.1) is also based on Zubov's theory. But in this case, the change of variables (4.9), (4.10) is more complicated.

I would like to thank A.P. Markeyev for suggesting the topic and for useful comments.

REFERENCES

1. MARKEYEV A.P. and SHCHERBINA G.A., The motions of a satellite asymptotic to its eccentric oscillations, Izv. Akad. Nauk SSSR, MTT, 4, 1987.
2. MARKEYEV A.P. and SHCHERBINA G.A., On motions asymptotic to triangular points of libration in the circular bounded three-body problem, PMM, 51, 3, 1987.
3. MARKEYEV A.P., Resonances and asymptotic trajectories in Hamiltonian systems, PMM, 54, 2, 1990.
4. IVANOV A.P. and SOKOL'SKII A.G., The stability of non-autonomous Hamiltonian system with ground-type parametric resonance, PMM, 44, 6, 1980.
5. LYAPUNOV A.M., General Problem of the Stability of Motion, Collected Papers. 2, Izv. Akad. Nauk SSSR, Moscow-Leningrad, 1956.
6. POINCARÉ H., New Methods of Celestial Mechanics, Collected Papers, 1, Nauka, Moscow, 1971.
7. BIRKHOFF G., Dynamic Systems, Gostekhizdat, Moscow-Leningrad, 1941.
8. NEMYTSKII V.V. and STEPANOV V.V., Qualitative Theory of Differential Equations, Gostekhizdat Moscow-Leningrad, 1949.
9. SOKOL'SKII A.G., On the stability of autonomous Hamiltonian system with two degrees of freedom for first-order resonance, PMM, 41, 1, 1977.
10. MARKEYEV A.P., Libration Points in Celestial Mechanics and Astrodynamics, Nauka, Moscow, 1978.
11. MOSER J., Lectures on Hamiltonian Systems, Mir, Moscow, 1972.
12. ZUBOV V.I. Stability of Motion, Vysshaya Shkola, Moscow, 1973.
13. MERMAN G.A., Asymptotic solutions of a canonical system with one degree of freedom with zero characteristic exponents, Byul. Inst. Teoret. Astron, 9, 6, 1964.
14. GIACAGLIA G.E.O., Methods of Pertrubation Theory for Non-linear Systems, Nauka, Moscow, 1979.
15. SOKOL'SKII A.G., Stability of Hamiltonian systems at zero frequency, Differ. Uravn., 8, 1981.

Translated by Z.L.